Calculating Complex Derivatives

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Vector Space

Definition

- A set ϑ of vectors is called a Vector Space if :
  - P1 $\mathbf{0} \in \vartheta$
  - P2 $\mathbf{x}, \mathbf{y} \in \vartheta \Rightarrow \mathbf{x} + \mathbf{y} \in \vartheta$
  - P3 $\mathbf{x} \in \vartheta \Rightarrow \forall k \in \Re, k\mathbf{x} \in \vartheta$
- All other properties of a Vector Space can be trivially derived

Vector Space

Vectors in $\Re^n$

A vector $\mathbf{x} \in \Re^n$ is of form $\mathbf{x} = \begin{bmatrix} k_1 \\ k_2 \\ . \\ . \\ . \\ k_{n-1} \\ k_n \end{bmatrix}$, where $k_1, k_2, k_3 \ldots k_n \in \Re$

Vector Space

Vectors in $\mathbb{M}_{m\times n}(\Re)$

- $\mathbb{M}_{m\times n}(\Re)$ is the set of all $m \times n$ matrices with real elements

- $\mathbb{M}_{m\times n}(\Re)$ forms a Vector Space because :

P1 $\mathbf{0}_{m\times n} \in \mathbb{M}_{m\times n}(\Re)$

P2 $\mathbf{x}_{m\times n}, \mathbf{y}_{m\times n} \in \mathbb{M}_{m\times n}(\Re) \Rightarrow \mathbf{x}_{m\times n} + \mathbf{y}_{m\times n} \in \mathbb{M}_{m\times n}(\Re)$

P3 $\mathbf{x}_{m\times n} \in \mathbb{M}_{m\times n}(\Re) \Rightarrow \forall k \in \Re, k\mathbf{x}_{m\times n} \in \mathbb{M}_{m\times n}(\Re)$

Basis
Intuition

- Basis of a Vector Space is used to define a coordinate system
- Basis of a Vector Space allows representation in $\Re^n$
- Basis is a set consisting of certain vectors in the Vector Space

Basis
Example

- Let $\mathbf{A} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \in \mathrm{M}_{2\times 2}(\Re)$ be a 2×2 matrix

- Let $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ be our basis

Basis

Example

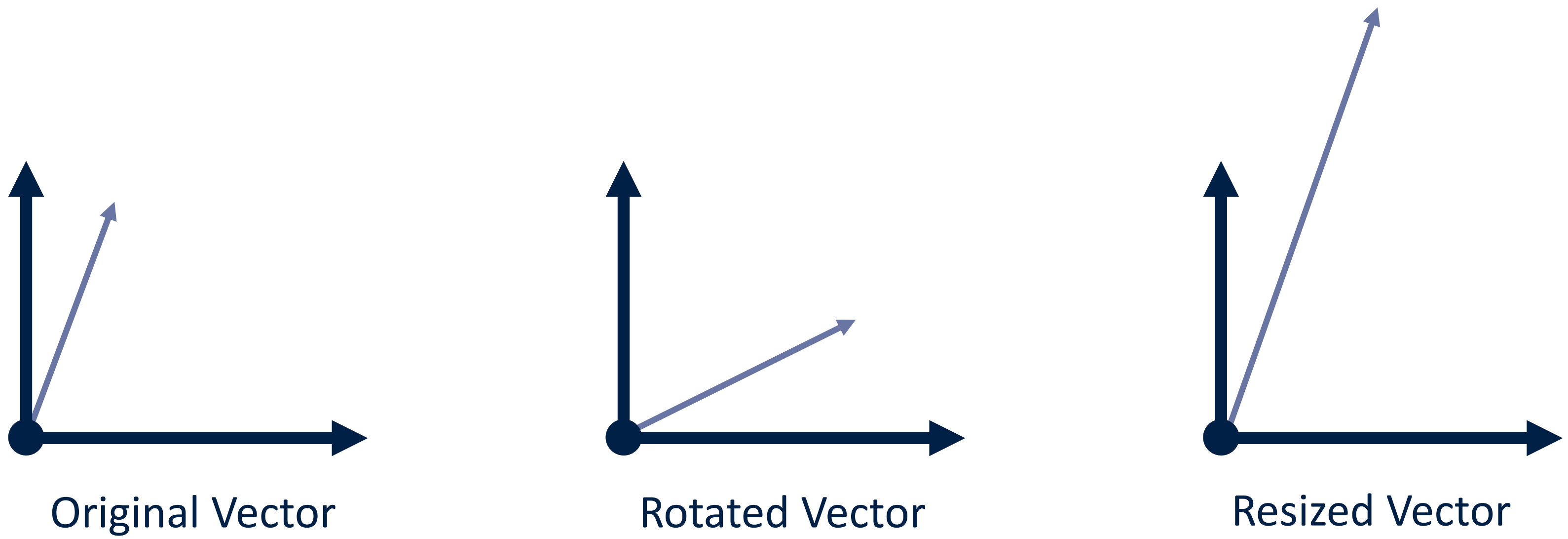
- $\mathbf{A} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} = k_{11}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k_{12}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + k_{21}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + k_{22}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

- $\mathbf{A}_{\beta} = \begin{bmatrix} k_{11} \\ k_{12} \\ k_{21} \\ k_{22} \end{bmatrix}$ is a representation of $\mathbf{A}$ in $\Re^4$

Linear Transformation

Definition

- An operation on a Vector that can be represented as multiplication with a matrix is called a Linear Transformation



Sine, Cosine Pair

Properties

- $a\sin(kx)+b\cos(kx)$ forms a Vector Space ϑ
- $\beta=\{\sin(\mathrm{kx}),\cos(\mathrm{kx})\}$ is a Basis for ϑ

$$\boxed{\forall \mathbf{y}\in\vartheta, \mathbf{y}=a\sin(kx)+b\cos(kx) \Rightarrow \mathbf{y}_{\beta}=\begin{bmatrix} a \\ b \end{bmatrix}_{\beta} \text{ in } \Re^{2}}$$

Sine, Cosine Pair

Derivative as a Linear Transformation

- $\forall \mathbf{y} \in \vartheta, \mathbf{y} = a\sin(kx) + b\cos(kx) \Rightarrow \mathbf{y}_\beta = \begin{bmatrix} a \\ b \end{bmatrix}_\beta$ in $\Re^2$

- $\mathbf{y}^{(1)} = ak\cos(kx) - bk\cos(kx) \Rightarrow \mathbf{y}^{(1)}{}_\beta = \begin{bmatrix} ak \\ -bk \end{bmatrix}_\beta$ in $\Re^2$

- Differentiating $y = a\sin(kx) + b\cos(kx)$ is a Linear Transformation because :

$$\mathbf{y}_\beta = \begin{bmatrix} a \\ b \end{bmatrix}_\beta \Rightarrow \mathbf{y}^{(1)}{}_\beta = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix} \mathbf{y}_\beta$$

Sine, Cosine Pair

Derivative as a Linear Transformation

By repeatedly differentiating,

$$\boxed{\mathbf{y}^{(1)}{}_{\beta} = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}^{1} \mathbf{y}_{\beta} \Rightarrow \mathbf{y}^{(m)}{}_{\beta} = \begin{bmatrix} k & 0 \\ 0 & -k \end{bmatrix}^{m} \mathbf{y}_{\beta}}$$

Fourier Series

Derivative as a Linear Transformation

- Using Fourier Series, any function $\mathrm{f}(x)$ that satisfies Dirchtlet's Conditions can be expressed as:

$$\mathrm{f}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\omega x) + b_n \sin(n\omega x))$$

- Let,

$$\beta_n = \{\cos(n\omega x), \sin(n\omega x)\}$$

$$\mathbf{y}_n = a_n \cos(n\omega x) + b_n \sin(n\omega x)$$

Thus, $\mathbf{y}_{\mathrm{n},\beta_n} = \begin{bmatrix} a_n \\ b_n \end{bmatrix}_{\beta_n}$ in $\Re^2$

Fourier Series

Derivative as a Linear Transformation

- For simplicity, ignore functions with $a_0 \neq 0$

$$\mathrm{f}(x) = \sum_{n=1}^{\infty} (a_n \cos(n\omega x) + b_n \sin(n\omega x))$$

$$\mathrm{f}_\beta(x) = \sum_{n=1}^{\infty} \mathbf{y}_{n,\beta_n} = \sum_{n=1}^{\infty} \begin{bmatrix} a_n \\ b_n \end{bmatrix}_{\beta_n}$$

- Calculating m^{th} derivative of $\mathrm{f}_\beta(x)$ as a Linear Transformation:

$$\mathrm{f}^{(m)}{}_\beta(x) = \sum_{n=1}^{\infty} \begin{bmatrix} n\omega & 0 \\ 0 & -n\omega \end{bmatrix}^m \begin{bmatrix} a_n \\ b_n \end{bmatrix}_{\beta_n}$$

Fourier Series

Derivative as a Linear Transformation

- Thus, it is possible to calculate m^{th} derivative of $\mathrm{f}_{\beta}(x)$ using :

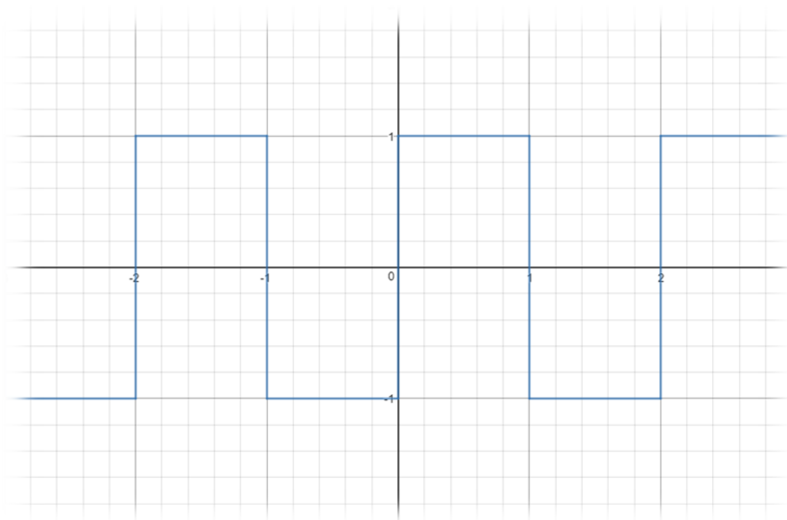
$$\mathrm{f}^{(m)}{}_{\beta}(x) = \sum_{n=1}^{\infty} \begin{bmatrix} n\omega & 0 \\ 0 & -n\omega \end{bmatrix}^{m} \begin{bmatrix} a_n \\ b_n \end{bmatrix}_{\beta_n}$$

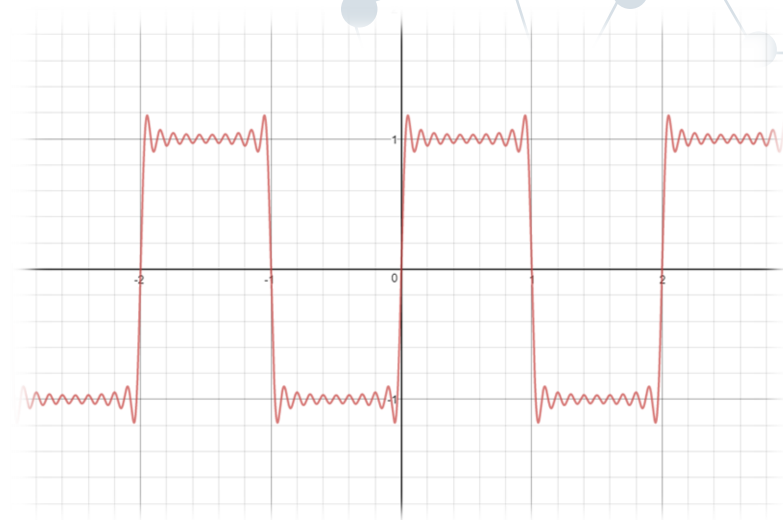
- What happens when we choose m to be a fraction, or an Imaginary Number?

$$\mathrm{f}^{(-1)}{}_{\beta}(x), \mathrm{f}^{(-0.5)}{}_{\beta}(x), \mathrm{f}^{(j)}{}_{\beta}(x)\ldots$$

Fourier Series

Square Wave





- Square Wave can be represented as:

$$\mathrm{f}_{\beta}(x) = \sum_{n=1}^{\infty} \begin{bmatrix} 0 \\ b_n \end{bmatrix}_{\beta_n} \text{, where } b_n = \left[\frac{2(1\text{-}(\text{-}1)^n)}{n\pi} \right]$$

Fourier Series

Derivative with $m = -1$

$$f^{(-1)}{}_{\beta}(x) = \sum_{n=1}^{\infty} \begin{bmatrix} \alpha & \beta n\pi \\ -\beta n\pi & \alpha \end{bmatrix} \begin{bmatrix} 0 \\ b_n \end{bmatrix}_{\beta_n}$$

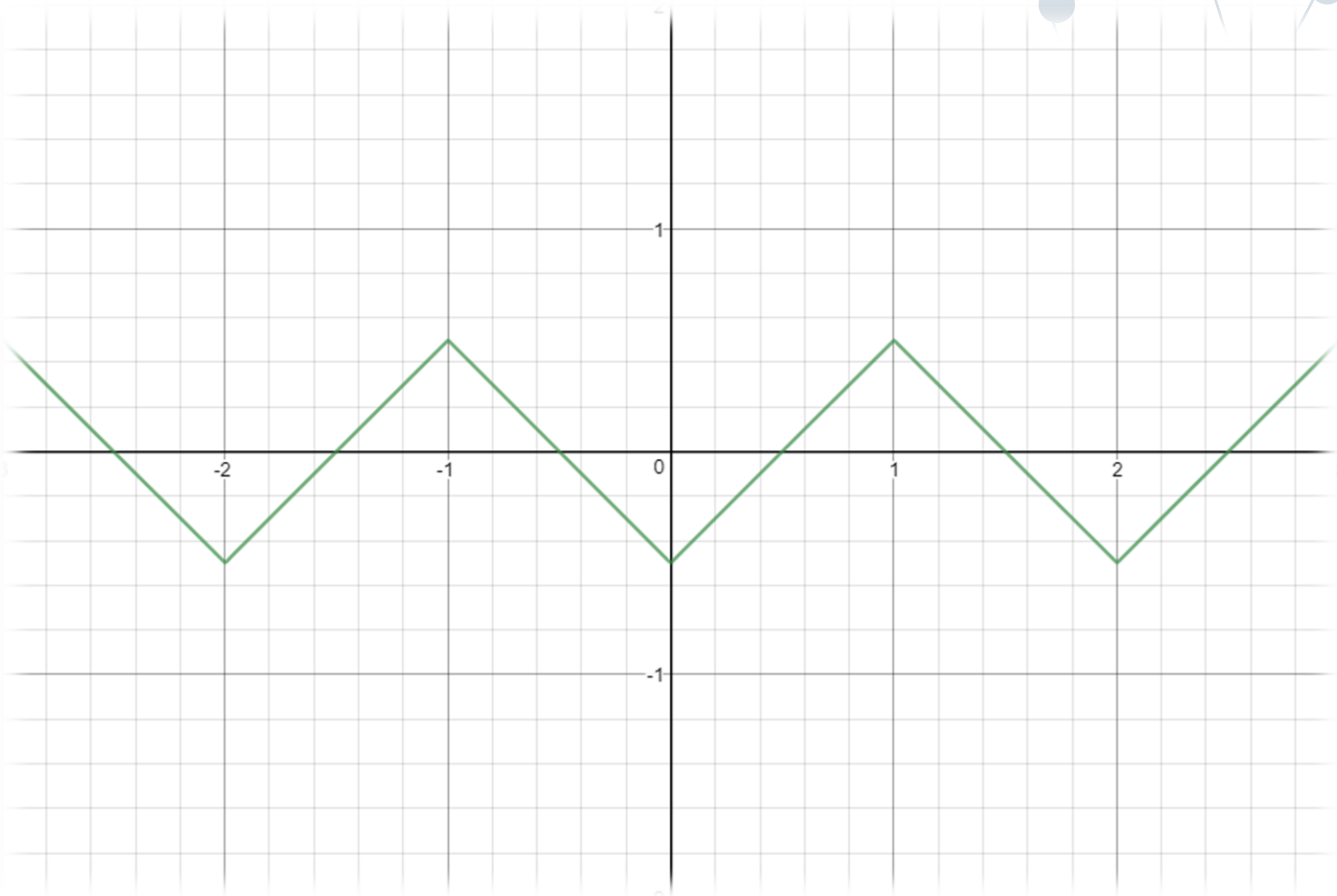
where,

$$b_n = \left[\frac{2(1\text{-}(\text{-}1)^n)}{n\pi} \right]$$

$$\alpha = 0, \beta = -\frac{1}{(n\pi)^2}$$

Fourier Series

Derivative with $m = -1$



Fourier Series

Derivative with $m = -0.5$

$$f^{(-0.5)}{}_{\beta}(x) = \sum_{n=1}^{\infty} \begin{bmatrix} \alpha & \beta n\pi \\ -\beta n\pi & \alpha \end{bmatrix} \begin{bmatrix} 0 \\ b_n \end{bmatrix}_{\beta_n}$$

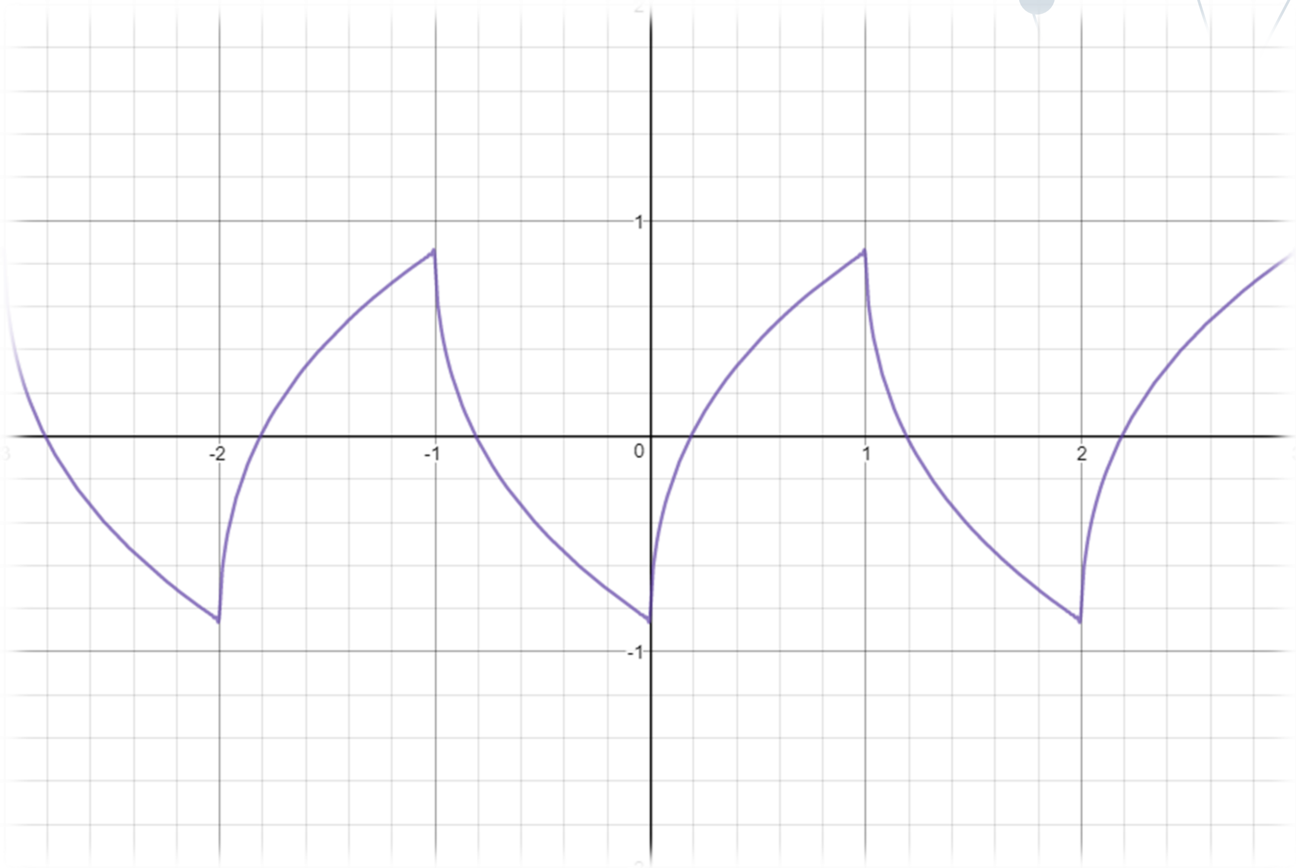
where,

$$b_n = \left[\frac{2(1-(-1)^n)}{n\pi} \right]$$

$$\alpha = \frac{1}{\sqrt{2n\pi}}, \beta = -\frac{1}{\sqrt{2}(n\pi)^{3/2}}$$

Fourier Series

Derivative with $m = -0.5$



Fourier Series

Derivative with $m = j = \sqrt{-1}$

$$\mathrm{f}^{(j)}{}_{\beta}(x) = \sum_{n=1}^{\infty} \begin{bmatrix} \alpha & \beta n\pi \\ -\beta n\pi & \alpha \end{bmatrix} \begin{bmatrix} 0 \\ b_n \end{bmatrix}_{\beta_n}$$

where,

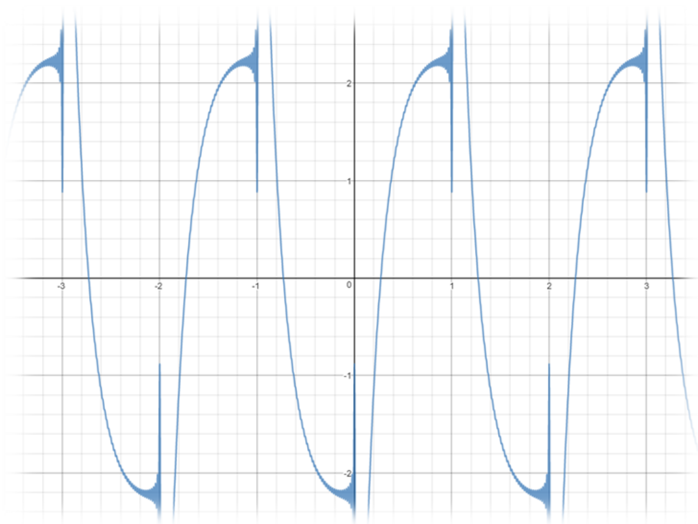
$$b_n = \left[\frac{2(1\text{-}(\text{-}1)^n)}{n\pi} \right]$$

$$\alpha = (n\pi)^j \cos\left(\frac{\pi}{2} j\right), \beta = (n\pi)^{j-1} \sin\left(\frac{\pi}{2} j\right)$$

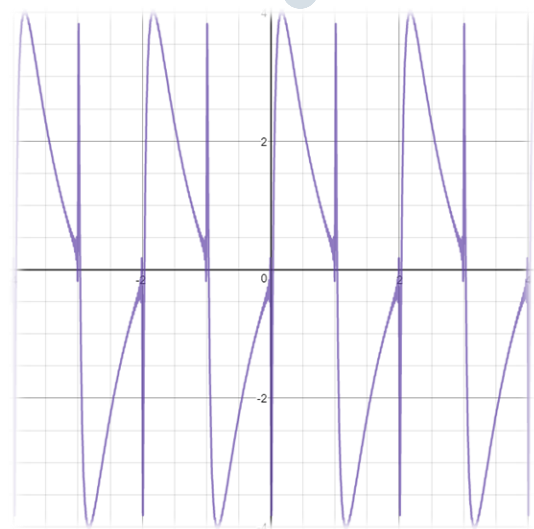
Fourier Series

Derivative with $m = j = \sqrt{-1}$

- $\mathrm{Re}(\mathrm{f}^{(j)}{}_{\beta}(x))$



- $\mathrm{Im}(\mathrm{f}^{(j)}{}_{\beta}(x))$



Fourier Series

Derivative with $m = -x^{-1}$

$$f^{(-x^{-1})}{}_{\beta}(x) = \sum_{n=1}^{\infty} \begin{bmatrix} \alpha & \beta n \pi \\ -\beta n \pi & \alpha \end{bmatrix} \begin{bmatrix} 0 \\ b_n \end{bmatrix}_{\beta_n}$$

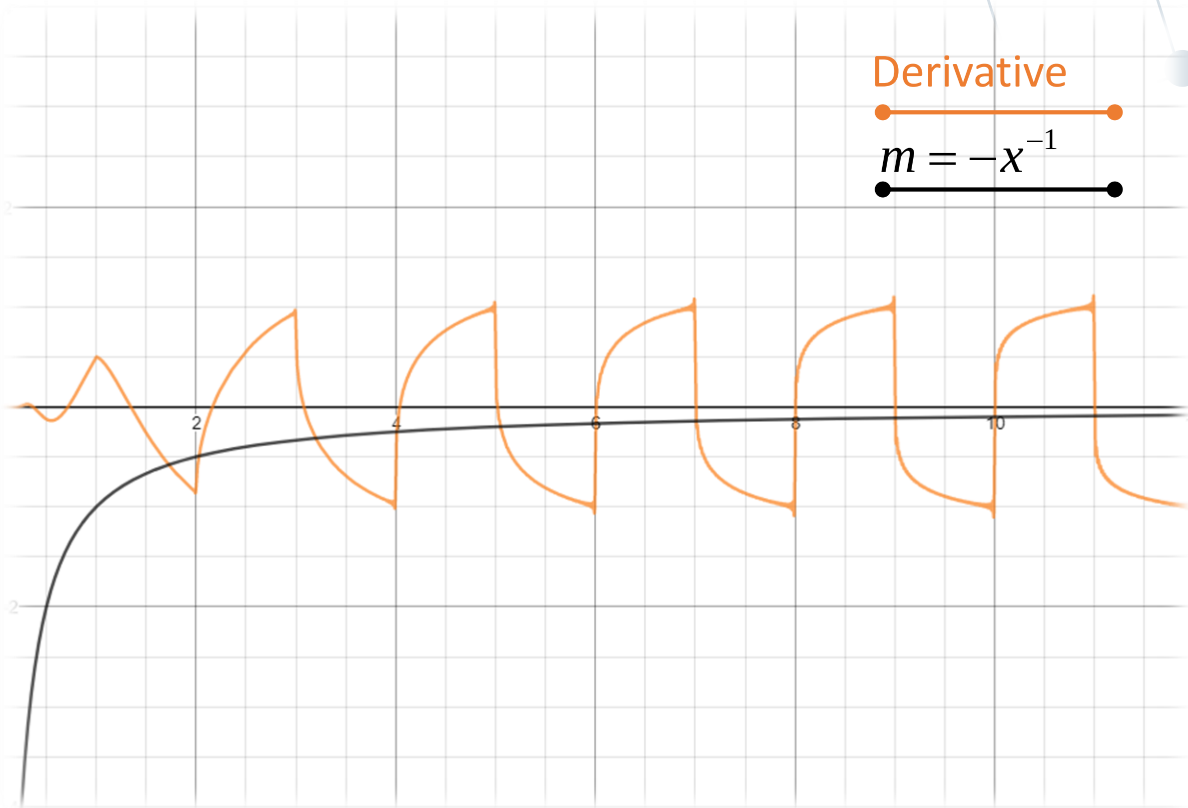
where,

$$b_n = \left[\frac{2(1-(-1)^n)}{n\pi} \right]$$

$$\alpha = (n\pi)^{-x^{-1}} \cos\left(\frac{\pi}{2} x^{-1} \right), \beta = -(n\pi)^{-x^{-1}-1} \sin\left(\frac{\pi}{2} x^{-1} \right)$$

Fourier Series

Derivative with $m = -x^{-1}$



Summary

- Derivative of $a\sin(kx)+b\cos(kx)$ is a Linear Transformation
- Derivative Matrix of $a\sin(kx)+b\cos(kx)$ allows calculation of complex derivatives
- We can *sometimes* calculate complex derivatives of certain functions using their Fourier Expansion

Questions?

References

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