# How topos theory can help algebra and geometry

*– an invitation –*

Ingo Blechschmidt
July 10th, 2018

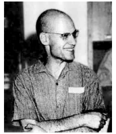


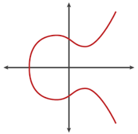
This talk has two parts. In the first part, we'll learn that besides the standard mathematical universe, in which ordinary mathematics takes place, there is a host of alternate mathematical universes. In these alternate universes, the usual objects of mathematics enjoy slightly different properties. For instance, we'll encounter universes in which the intermediate value theorem fails or in which any map $\mathbb{R} \to \mathbb{R}$ is continuous.

In the second part, we'll see that these alternate universes, while seeming strange on first contact, yield concrete applications in algebra and geometry.

# Toposes are …

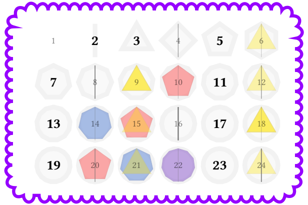
## generalized spaces





étale topos of a scheme
field with one element

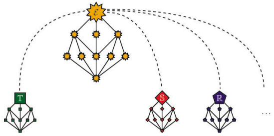
## mathematical universes




## categories of sheaves

A topos is a finitely complete cartesian closed category with a subobject classifier.

## embodiments of theories



“Let $G$ be a group.”



Toposes were invented in the 1960s by Grothendieck in order to solve concrete problems in algebraic geometry. Their raison d’être is the following. In algebraic geometry, we want (and sometimes have to) work over fields other than $\mathbb{R}$ and $\mathbb{C}$ and even over arbitrary commutative rings. For the geometric objects in such settings, the *schemes*, the Euclidean topology is not available; we have to make do with the *Zariski topology*. However, important tools as singular cohomology don’t work well with this topology (too few opens).

The problem was solved by inventing *étale topology* as an enhancement of the Zariski topology. However, contrary to its name, the étale topology isn’t actually a topology in the usual sense. Putting the étale topology on a scheme doesn’t yield a refined topological space, but a *topos*.

Toposes generalize topological spaces in two ways: Firstly, the “open sets” of toposes don’t actually have to be sets of points; they can be more general kinds of objects such as coverings. Toposes can even have no points at all and still be nontrivial (this is for instance the case for the *topos of random sequences*). Secondly, while classically a given open subset is either contained in a further open subset or not, the opens of toposes can be contained in further opens in many different ways.

Recently, toposes are being used to help the mythical field with one element come into being.

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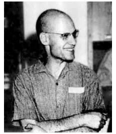

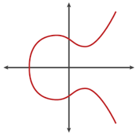

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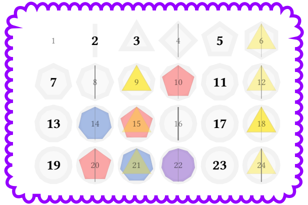


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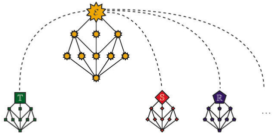


"Let $G$ be a group."



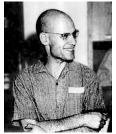
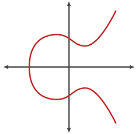
Since the 1960s, many more aspects of toposes were discovered. A popular reference on toposes starts with a list of 13 ways of viewing toposes.

In this talk, we'll focus on the view of toposes as *alternate mathematical universes*. We can do mathematics inside these alternate universes just as well as we can do mathematics inside the *standard universe* (which is represented by a particular topos called "Set"). Each topos contains own versions of all the familiar mathematical objects – numbers, functions, manifolds – but the properties they enjoy can differ slightly from the properties they enjoy in the standard topos.

The definition of what a topos is, displayed in the lower left of the slide, has two problems. Firstly, it's only useful if one knows the relevant category-theoretic jargon. Secondly, a topos has lots of further vital structure, which is crucial for a rounded understanding, but not listed in the displayed definition (which is trimmed for minimality). A more comprehensive definition is: A *topos* is a locally cartesian closed, finitely complete and cocomplete Heyting category which is exact, extensive and has a subobject classifier. We won't need either definition in this talk.

# Toposes are …

## generalized spaces

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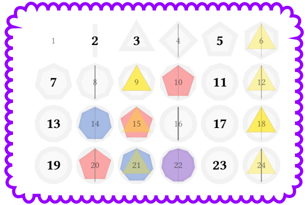


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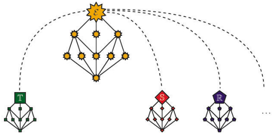
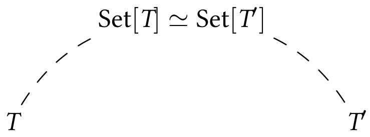


“Let $G$ be a group.”



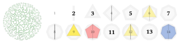
Toposes can also be thought as reifying the “semantic essence” of mathematical theories (the theory of groups, the theory of rings, the theory of intervals, …). Given any such theory $T$, there is a so-called *classifying topos* $\mathrm{Set}[T]$. Its points are precisely the set-based models of $T$ (the groups, the rings, the intervals, …), and it contains a *generic model* which has exactly those properties which all models have. (This generic model is what mathematicians implicitly refer to when they say “Let $G$ be a group.”.)

Crucially, two theories $T$ and $T'$ can have equivalent classifying toposes even when they are not syntactically related in any way. This observation is the starting point of Olivia Caramello’s *bridge technique*, a grand research program with applications in many different fields.



# A glimpse of the toposophic landscape

**Set**



The usual laws of logic hold.



The topos Set is the standard topos. This topos is where ordinary mathematics takes place.

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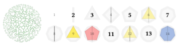

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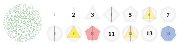
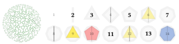
**Sh $X$**



The intermediate value theorem fails.

**Eff**



Every function is computable.



Besides Set, there is a proper class' worth of further toposes. We'll get to know some of these better during the course of the talk.

For any topological space $X$, there is the topos $\mathrm{Sh}(X)$ of *sheaves over* $X$. They are useful in analysis for internalizing parameter-dependence. Apart from pathological cases like $X$ being discrete, the intermediate value theorem in the form

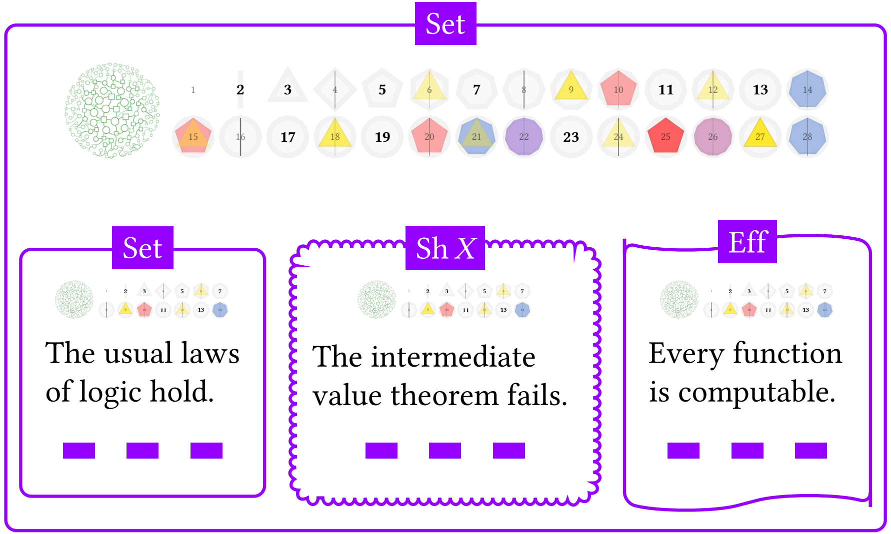
> *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume $f(-1) < 0 < f(1)$. Then there is a number $x \in \mathbb{R}$ such that $f(x) = 0$.*

fails in these toposes – for very meaningful reasons discussed below. The following, classically equivalent, version does hold:

> *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume $f(-1) < 0 < f(1)$. Then, for every $\varepsilon > 0$, there is a number $x \in \mathbb{R}$ such that $|f(x)| < \varepsilon$.*

The *effective topos* Eff is a computer scientist's dream come true: In it, any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine. The effective topos and its close cousins can be used to study the differences between the many models of computation, particularly those differences which are only visible at higher types.

# A glimpse of the toposophic landscape






This talk is still part of ordinary mathematics, that is, of the topos Set. Therefore a more accurate picture depicts the three examples as part of Set. (Technically, while they might not be "toposes over Set", they are still locally-internal to Set in the sense of Penon.)

Most of the toposes in active use are either toposes of sheaves (over a topological space or a *site*), realizability toposes (such as Eff or variants constructed using different models of computation), or arise from those using topos-theoretic constructions; but this is not a complete classification. (The topos Set is equivalent to the topos of sheaves over the one-point space.)

# The internal universe of a topos

For any topos $\mathcal{E}$ and any statement $\varphi$, we define the meaning of

"$\mathcal{E} \models \varphi$" ("$\varphi$ holds in the internal universe of $\mathcal{E}$")

using the **Kripke–Joyal semantics**.

| $\mathrm{Set} \models \varphi$ | $\mathrm{Sh}(X) \models \varphi$ | $\mathrm{Eff} \models \varphi$ |
|---|---|---|
| "$\varphi$ holds in the usual sense." | "$\varphi$ holds continuously." | "$\varphi$ holds computably." |



We can picture the Kripke–Joyal semantics as a kind of translation engine. When we're talking over the phone with Anna, a mathematician who lives in the effective topos, at first we might feel uncomfortable when she states "it's a basic fact of life that any function $\mathbb{N} \to \mathbb{N}$ is computable". But if we remember to switch on the Kripke–Joyal translation, we instead hear "it's a basic fact of life that there is a Turing machine which, given a Turing machine computing a function $f : \mathbb{N} \to \mathbb{N}$, outputs a Turing machine computing $f$" which we can easily agree with.

The precise translation rules will be explained by osmosis for the effective topos, on the next slide, and by a formal definition for sheaf toposes and for the little Zariski topos, further below.

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Any topos supports **mathematical reasoning**:

If $\mathcal{E} \models \varphi$ and if $\varphi$ entails $\psi$ constructively, then $\mathcal{E} \models \psi$.



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When exploring a new topos for the first time, the only way to find out which statements hold in it is to translate them using the Kripke–Joyal semantics and check whether the translation holds in the usual mathematical sense. As soon as we have established a certain stock of statements in this way, we can switch to a more efficient procedure: We can just use mathematical reasoning to deduce new statements from known ones.

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no $\varphi \vee \neg\varphi$, no $\neg\neg\varphi \Rightarrow \varphi$, no axiom of choice



Irrespective of philosophical preferences, it's a fact of life that most toposes only support constructive reasoning; in most toposes, proof by contradiction is not valid. (Examples for toposes in which this is possible are sheaf toposes $\mathrm{Sh}(X)$ over discrete topological spaces, but not many more than that.)

One might fear that most of mathematics breaks down in a constructive setting. This is only true if interpreted naively: Often, already very small changes to the definitions and statements suffice to make them constructively valid (and are classically simply equivalent reformulations). In other cases, we need to add interesting additional hypotheses – hypotheses which are classically always satisfied. Here are a couple of examples.

1. The usual proof that $\sqrt{2}$ is not rational is perfectly fine from a constructive point of view. It shows that the assumption that $\sqrt{2}$ is rational entails a contradiction. This is just the definition of what it means to be not rational. (There's a difference between a proof by contradiction and a proof of a negated statement. Only the former can't generally be carried out in constructive mathematics.)

2. Constructively it's still true that there is no such thing as a statement which is neither true nor false: That is, we still have $\neg\neg(\varphi \vee \neg\varphi)$.

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3. Constructively, we can't show that any inhabited subset of the natural numbers has a minimal element. [We can also not show the negation of that statement – any valid constructive proof is a fortiori a valid classical proof.] But we can show (quite easily, by induction) that any inhabited and *detachable* subset of the natural numbers has a minimal element: A subset $U \subseteq \mathbb{N}$ is detachable iff for any number $n \in \mathbb{N}$, $n \in U$ or $n \notin U$. Weakening the conclusion, we can also show that any inhabited subset of the natural numbers does *not not* have a minimal element.

   Both the failure and the two fixes can be interpreted computationally: Given just the promise of an inhabited subset, we can't algorithmically determine its minimum. But we can do so when given a *membership oracle*, or if it's okay to return the result in the *continuation monad*.

4. We can't constructively prove that any finitely generated vector space admits a basis. We can, however, constructively verify that any finitely generated vector does *not not* admit a basis, (By exploiting that given a generating family $(x_1, \ldots, x_n)$, it's *not not* the case that either one of the generators is a linear combination of the others, or not.) We'll see below that this particular example entails that any sheaf of finite type over a reduced scheme is locally free on a dense open.

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5. We can't constructively verify the fundamental theorem of Galois theory for arbitrary (not necessarily finite) Galois extensions in its usual formulation. But if we pass from the topological Galois group to the *localic Galois group*, we can. Some aspects of the proof even get simpler that way.

6. Similarly, we can't constructively verify the Gelfand–Neumark correspondence between commutative $C^*$-algebras with unit and compact Hausdorff spaces. But we can do so if we pass from compact Hausdorff spaces to compact Hausdorff locales.

A recommendation for more on constructive mathematics is the informative and entertaining talk *Five Stages of Accepting Constructive Mathematics* by Andrej Bauer (video, notes).

# First steps in alternate universes

- Eff $\models$ "Any number is prime or is not prime." ✓
  Meaning: There is a **Turing machine** which determines of any given number whether it is prime or not.

- Eff $\models$ "There are infinitely many prime numbers." ✓
  Meaning: There is a **Turing machine** producing arbitrarily many primes.

- Eff $\models$ "Any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not." ✗
  Meaning: There is a **Turing machine** which, given a Turing machine computing a function $f : \mathbb{N} \to \mathbb{N}$, determines whether $f$ is zero or not.

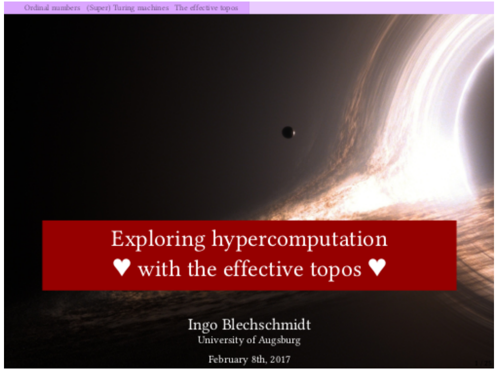
- Eff $\models$ "Any function $\mathbb{N} \to \mathbb{N}$ is computable." ✓

- Eff $\models$ "Any function $\mathbb{R} \to \mathbb{R}$ is continuous." ✓

- Sh($X$) $\models$ "Any cont. function with opposite signs has a zero." ✗
  Meaning: Zeros can locally be picked **continuously** in continuous families of continuous functions. (video for counterexample)



There is a variant of the effective topos which is not built using Turing machines, but using *infinite-time Turing machines*, a popular model for hypercomputation. In that variant, the statement "any function $\mathbb{N} \to \mathbb{N}$ is the zero function or not" is true; the statement "any function $\mathbb{N} \to \mathbb{N}$ is computable by a Turing machine" is false; and the statement "any function $\mathbb{N} \to \mathbb{N}$ is computable by an infinite-time Turing machine" is true again. Details can be found in this set of slides.



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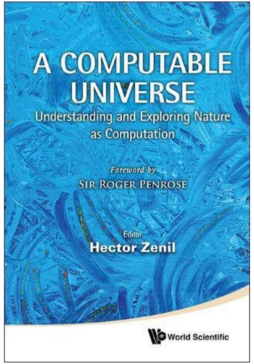
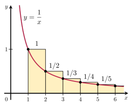
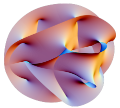
We can also build a variant of the effective topos which uses *machines of the real world* instead of idealized Turing machines. In doing so, we leave the realm of rigorous mathematics, but obtain interesting connections with philosophy and physics. For instance, in that variant the statement "any function $\mathbb{R} \to \mathbb{R}$ is continuous" is true if machines in the real world can only perform finitely many computational steps in finite time and if it's possible to build hidden communication channels. Details can be found in the book chapter *Intuitionistic Mathematics and Realizability in the Physical World* by Andrej Bauer.



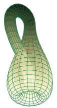
# Applications



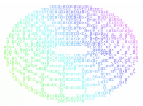
**analysis**
internalizing
parameter-dependence



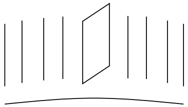
**algebraic geometry**
reducing geometry to algebra
reducing relative to absolute
synthetic account



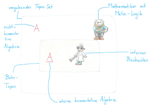
**differential geometry**
reflection principles
synthetic account



**homotopy theory**
synthetic account
computer-assisted proofs
generalizations



**commutative algebra**
local-to-global principles
reduction techniques
constructive proofs



**further subjects**
synth. computability th.
synth. measure theory
Bohr topos for QM



The applications of the internal language of toposes in several branches of mathematics are driven by the following fundamental empirical fact:

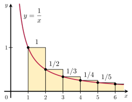
The external translations of
some easy-to-prove internal statements of suitable toposes
are
nontrivial statements about the subject matter at hand.

For instance, Grothendieck's generic freeness lemma, an important theorem in commutative algebra and algebraic geometry, is the external translation of the basic observation "finitely generated modules over finitely generated algebras over fields are *not not* free over the field and *not not* finitely presented over the algebra" of constructive linear algebra.

Is there a theorem which can only be proven by topos-theoretic methods? *No,* for any proof employing the internal language can be unwound to yield an external proof not referencing topos theory. Just as the translation from internal statements to external ones is entirely mechanical, so is the translation from internal proofs to external ones. However, this translation will always make the proof longer, and, depending on the syntactical complexity of the involved statements, the resulting external proof might be quite complex.

The next two slides show examples regarding this increase in complexity.

# Applications



**analysis**
internalizing
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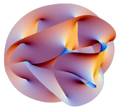
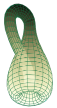
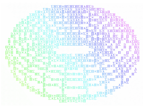


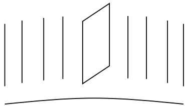
**algebraic geometry**
reducing geometry to algebra
reducing relative to absolute
synthetic account



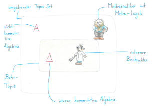
**differential geometry**
reflection principles
synthetic account



**homotopy theory**
synthetic account
computer-assisted proofs
generalizations



**commutative algebra**
local-to-global principles
reduction techniques
constructive proofs



**further subjects**
synth. computability th.
synth. measure theory
Bohr topos for QM

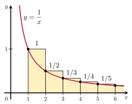


1. The statement "if the two outer sheaves in a short exact sequence of sheaves of modules are of finite type, then so is the middle one" is the external translation of the basic fact "if the two outer modules in a short exact sequence of modules are finitely generated, then so is the middle one" of constructive linear algebra.

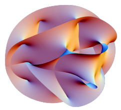
   The syntactical complexity of the internal statement is quite low (just one implication sign, at toplevel, no long chains of quantifiers of mixed types). Therefore the resulting external proof obtained by unwinding will still be quite short and straightforward. In fact, this external proof will coincide with what anyone well-versed in scheme theory will produce, judiciously juggling open subsets.

   Using the internal language of toposes in situations like this is therefore mostly for mental hygiene, making rigorous the intuitive idea that the two statements are related to each other.

# Applications



**analysis**
internalizing
parameter-dependence



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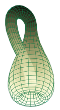

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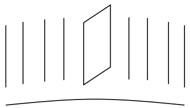

**commutative algebra**
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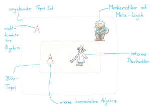


**further subjects**
synth. computability th.
synth. measure theory
Bohr topos for QM

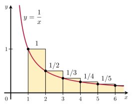


2. A version of Hilbert's basis theorem, "polynomial rings over anonymously Noetherian rings are again anonymously Noetherian", can be put to use in the proof of Grothendieck's generic freeness lemma alluded to above.

   This statement has nontrivial syntactical complexity (the Noetherian condition involves a double negation, therefore nested implications). For this reason, already the translation of the *statement* is quite convoluted (a part of it is displayed in this set of slides for your viewing pleasure). The translation of its *proof* contains a deformed copy of the standard proof of Hilbert's basis theorem (in a way which doesn't allow to simply *apply* Hilbert's basis theorem, one has to actually redo its proof).

   In situations like this, the internal language is of real value.

# Applications

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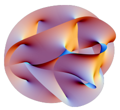

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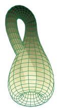

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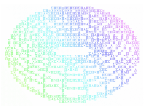

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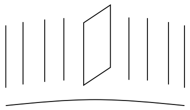

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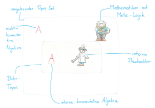

**further subjects**
synth. computability th.
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We're used to the fact that the usual laws of mathematical reasoning apply to mathematical objects. The internal language of toposes encapsulates and makes useful the following observation: The usual laws of reasoning apply to our favorite mathematical objects *also in nontrivial ways different from the accustomed one*. I find this quite astonishing.

The next slide shows a basic application of this train of thought in analysis. The presented example is of a rather simple nature and serves only to explain the translation process in an explicit and rigorous fashion, contrasting the previous slides which were somewhat short on details. Details on the application to homotopy theory can be found in the HoTT book.

A tour of applications in algebraic geometry can be found in these notes.

# The topos of sheaves over a space

Let $X$ be a topological space. We recursively define

$$U \models \varphi \quad (\text{“}\varphi \text{ holds on } U\text{”})$$

for open subsets $U \subseteq X$ and statements $\varphi$. Write “$\mathrm{Sh}(X) \models \varphi$” to mean $X \models \varphi$. Let $\mathcal{C}(U)$ be the set of continuous functions $U \to \mathbb{R}$.

| | | |
|---|---|---|
| $U \models \top$ | iff | true |
| $U \models \bot$ | iff | ~~false~~ $U = \emptyset$ |
| $U \models s = t$ | iff | $s(x) = t(x)$ for all $x \in U$ |
| $U \models \varphi \wedge \psi$ | iff | $U \models \varphi$ and $U \models \psi$ |
| $U \models \varphi \vee \psi$ | iff | ~~$U \models \varphi$ or $U \models \psi$~~ there exists an open covering $U = \bigcup_i U_i$ such that for all $i$: $U_i \models \varphi$ or $U_i \models \psi$ |
| $U \models \varphi \Rightarrow \psi$ | iff | for all open $V \subseteq U$: $V \models \varphi$ implies $V \models \psi$ |
| $U \models \forall s : \mathbb{R}.\, \varphi(s)$ | iff | for all open $V \subseteq U$ and functions $s_0 \in \mathcal{C}(V)$: $V \models \varphi(s_0)$ |
| $U \models \exists s : \mathbb{R}.\, \varphi(s)$ | iff | ~~there exists $s_0 \in \mathcal{C}(U)$ such that $U \models \varphi(s_0)$~~ there exists an open covering $U = \bigcup_i U_i$ such that for all $i$: there exists $s_0 \in \mathcal{C}(U_i)$ such that $U_i \models \varphi(s_0)$ |



It’s an instructive exercise to verify that $\mathrm{Sh}(X) \models \neg\neg\varphi$ if and only if there is a dense open subset $U \subseteq X$ such that $U \models \varphi$. This equivalence gives geometric meaning to the failure of classical logic in $\mathrm{Sh}(X)$.

# Internalizing parameter-dependence

Let $(f_x)_{x \in X}$ be a continuous family of continuous functions (that is, let a continuous function $X \times \mathbb{R} \to \mathbb{R}$, $(x, a) \mapsto f_x(a)$ be given). From the internal point of view of $\mathrm{Sh}(X)$, this family looks like a single function $f : \mathbb{R} \to \mathbb{R}$.

- $\mathrm{Sh}(X) \models$ (The function $f : \mathbb{R} \to \mathbb{R}$ is continuous).
- Iff $f_x(-1) < 0$ for all $x \in X$, then $\mathrm{Sh}(X) \models f(-1) < 0$.
- Iff $f_x(+1) > 0$ for all $x \in X$, then $\mathrm{Sh}(X) \models f(+1) > 0$.
- Iff all $f_x$ are increasing, then $\mathrm{Sh}(X) \models$ ($f$ is increasing).
- Iff there is an open cover $X = \bigcup_i U_i$ such that for each $i$, there is a continuous function $s : U_i \to \mathbb{R}$ with $f_x(s(x)) = 0$ for all $x \in U_i$, then $\mathrm{Sh}(X) \models \exists s : \mathbb{R}.\, f(s) = 0$.



Hence:

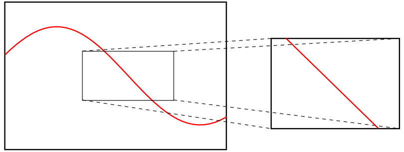
1. The standard formulation of the intermediate value theorem fails in $\mathrm{Sh}(X)$, because its external interpretation is that in continuous families of continuous functions, zeros can locally be picked continuously. That claim is false, as this counterexample demonstrates.

   As a corollary, we deduce that the standard formulation of the intermediate value theorem is not constructively provable.

2. The approximative version of the intermediate value theorem (stating that for any $\varepsilon > 0$, there is a number $x$ such that $|f(x)| < \varepsilon$) has a constructive proof and therefore holds in $\mathrm{Sh}(X)$. The external interpretation is that in continuous families of continuous functions, approximate zeros can locally be picked continuously.

3. The monotone intermediate value theorem, stating that a strictly increasing continuous function with opposite signs has a unique zero, admits a constructive proof and therefore holds in $\mathrm{Sh}(X)$. The external interpretation is that in continuous families of strictly increasing continuous functions, zeros can globally be picked continuously. You are invited to prove this fact directly, without reference to the internal language. This exercise isn't particularly hard, but it's not trivial either.

# Synthetic differential geometry



**The axiom of microaffinity**

Let $\Delta = \{\varepsilon \in \mathbb{R} \mid \varepsilon^2 = 0\}$. For any function $f\colon \Delta \to \mathbb{R}$, there is a unique number $a \in \mathbb{R}$ such that $f(\varepsilon) = f(0) + a\varepsilon$ for all $\varepsilon \in \Delta$.

- The **derivative** of a function $f\colon \mathbb{R} \to \mathbb{R}$ at $x_0 \in \mathbb{R}$ is the unique number $a \in \mathbb{R}$ such that $f(x_0 + \varepsilon) = f(x_0) + a\varepsilon$ for all $\varepsilon \in \Delta$.
- Manifolds are **just sets**.
- A **tangent vector** to $M$ is a map $\Delta \to M$.



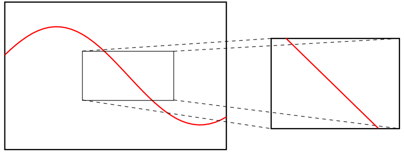
Differential geometry abounds with intuition which can't be formalized in the standard account of differential geometry. For instance, we can't literally define a tangent vector to be an infinitesimal piece of a curve – we have to resort to germs of smooth functions or derivations – or speak about infinitesimal deformations of the identity transformation.

Synthetic differential geometry was born to provide an account of differential geometry which is both rigorous and closer to intuition than the standard account, allowing for instance to read the classic works of Sophus Lie in a literal manner.

Its starting point is the axiom of microaffinity, which is quite useful but wildly false in classical mathematics. The fundamental theorem about synthetic differential geometry, connecting it with the ordinary world of smooth manifolds, states that there exist *well-adapted models* for synthetic differential geometry – toposes $\mathcal{E}$ such that:

1. There is a functorial way of associating to any smooth manifold $M$ an object $y(M)$ of $\mathcal{E}$ (something which $\mathcal{E}$ believes to be a set) and to any smooth map $f : M \to N$ a morphism $y(f) : y(M) \to y(N)$ of $\mathcal{E}$ (something which $\mathcal{E}$ believes to be map).
2. Any morphism of $\mathcal{E}$ of type $y(M) \to y(N)$ is of the form $y(f)$ for a smooth map $f$, and if $\mathcal{E} \models y(f) = y(g)$, then actually $f = g$.
3. In $\mathcal{E}$, the axiom of microaffinity and related axioms hold for the ring $y(\mathbb{R}^1)$. [Technical comment: This ring will usually not be the ring of Dedekind real numbers in $\mathcal{E}$.]

# Synthetic differential geometry



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- Manifolds are **just sets**.
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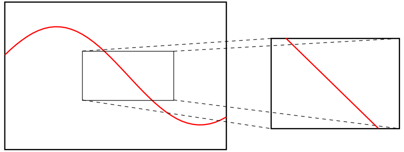
Here are two examples for synthetic reasoning and synthetic constructions.

- We can compute the derivative of $f$ with $f(x) = x^2$ as follows: For all $\varepsilon \in \Delta$, $f(x+\varepsilon) = (x+\varepsilon)^2 = x^2 + 2x\varepsilon$, hence $f'(x) = 2x$ by the definition on the slide.
- Write "$M^\Delta$" for the set of all maps $\Delta \to M$. A *vector field* on a manifold (set) $M$ is just a map $X : M \to M^\Delta$ such that $X(p)(0) = p$ for all $p \in M$. A vector field $X$ induces an infinitesimal path $\gamma : \Delta \to M^M$ in the space (set) of transformations of $M$, by setting $\gamma(\varepsilon) = (p \mapsto X(p)(\varepsilon))$.

  Conversely, a path $\gamma : \Delta \to M^M$ such that $\gamma(0) = \mathrm{id}_M$ yields a vector field $X : M \to M^\Delta$ by setting $X(p) = (\varepsilon \mapsto \gamma(\varepsilon)(p))$. No continuity or smoothness checks have to be carried out.

There are also notions in synthetic differential geometry which don't have a classical counterpart. For instance, in synthetic differential geometry it's possible to view differential forms on $M$ – which are usually thought of as *functionals* on (exterior powers of) the tangent bundle – as *quantities* (maps from $M$ into a certain nonclassical object).

# Synthetic differential geometry



**The axiom of microaffinity**

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- Manifolds are **just sets**.
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The following reflection principle is routinely used in undergraduate calculus: Even though continuity is officially a property which depends only on the input/output behaviour of a function, reflecting on the *term* used to define a given function often allows us to quickly conclude that it's continuous. For instance, any function defined by a polynomial expression is continuous.

Synthetic differential geometry yields a new reflection principle, for ordinary manifolds, which improves both on the assumption and on the conclusion: Any map between manifolds which is definable in constructive mathematics (so in particular, any map which is given by a polynomial expression) is smooth (so in particular continuous). This is due to item 2 above.

Synthetic differential geometry is very well-developed and at your disposal. References include:

- *Synthetische Differentialgeometrie*, notes for high school students (in German)
- *Intuitionistic mathematics for physics*, a blog post by Andrej Bauer
- *A Primer of Infinitesimal Analysis*, a short introduction by John Bell
- *Synthetic Differential Geometry*, the definitive book by Anders Kock

# The little Zariski topos of a ring

Let $A$ be a ring (commutative, with unit). We recursively define

$$D(f) \models \varphi \quad (\text{“}\varphi \text{ holds away from the zeros of } f\text{”})$$

for $f \in A$ and statements $\varphi$. Write “$\mathrm{Spec}(A) \models \varphi$” to mean $D(1) \models \varphi$.

| | |
|---|---|
| $D(f) \models \top$ | iff true |
| $D(f) \models \bot$ | iff $f$ is nilpotent |
| $D(f) \models x = y$ | iff $x = y \in M[f^{-1}]$ |
| $D(f) \models \varphi \wedge \psi$ | iff $D(f) \models \varphi$ and $D(f) \models \psi$ |
| $D(f) \models \varphi \vee \psi$ | iff there exists a partition $f^n = fg_1 + \cdots + fg_m$ with, for each $i$, $D(fg_i) \models \varphi$ or $D(fg_i) \models \psi$ |
| $D(f) \models \varphi \Rightarrow \psi$ | iff for all $g \in A$, $D(fg) \models \varphi$ implies $D(fg) \models \psi$ |
| $D(f) \models \forall x\colon M^\sim.\, \varphi(x)$ | iff for all $g \in A$ and $x_0 \in M[(fg)^{-1}]$, $D(fg) \models \varphi(x_0)$ |
| $D(f) \models \exists x\colon M^\sim.\, \varphi(x)$ | iff there exists a partition $f^n = fg_1 + \cdots + fg_m$ with, for each $i$, $D(fg_i) \models \varphi(x_0)$ for some $x_0 \in M[(fg_i)^{-1}]$ |



Irrespective of whether $A$ is a local ring, its mirror image $A^\sim$ is always a local ring (that is, the axioms of what it means to be a local ring hold in $\mathrm{Spec}(A)$).

A basic application of the internal language of $\mathrm{Spec}(A)$ are local-to-global principles. For instance:

- The statement “the kernel of a surjective matrix over a local ring is finite free” admits a constructive proof. It therefore holds in $\mathrm{Spec}(A)$. Its external meaning is that the kernel of a surjective matrix $M$ over $A$ is finite locally free (there exists a partition $1 = f_1 + \cdots + f_n$ such that for each $i$, the localized module $(\ker M)[f_i^{-1}]$ is finite free).
- The ring $A$ is a Prüfer domain if and only if $A^\sim$ is a Bézout domain. Therefore any constructive theorem about Bézout domains entails a corresponding theorem about Prüfer domains. Bézout domains are quite rare, while Prüfer domains abound (for instance the ring of integers of any number field is a Prüfer domain, even constructively so).

There are general metatheorems which state a precise connection between $A$ and $A^\sim$, allowing in some cases to quickly pass from one to the other.

# The little Zariski topos of a ring

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More advanced applications are rendered possible by the observation that, if $A$ is a reduced ring ($x^n = 0 \Rightarrow x = 0$), its mirror image $A^\sim$ is *anonymously Noetherian* (every ideal is *not not* finitely generated) and *a field*. This fact doesn't have a classical counterpart – in general, neither $A$ nor its stalks nor its quotients nor its subrings are Noetherian or fields.

This reduction technique allows to give a new proof of Grothendieck's generic freeness lemma which substantially improves on the previously known proofs in length and clarity: from approximately three pages (requiring several advanced prerequisites in commutative algebra) to a single paragraph (requiring only the tiny bit of topos theory needed in order to setup the internal language).

Sometimes, the effects of this topos-theoretic reduction technique can be mimicked by classical techniques in commutative algebra such as passing to quotient rings or stalks. To the best of my knowledge, that's not the case for Grothendieck's generic freeness lemma. (In other cases, where they can, the classical techniques require the axiom of choice, while the topos-theoretic technique doesn't and therefore yields a more calculational, informative proof.)

Details can be found in this set of slides.

# The little Zariski topos of a ring

Let $A$ be a ring (commutative, with unit). We recursively define

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As mentioned before, any proof involving the internal language can be *unwound* to yield a direct proof not referencing toposes. Depending on the logical complexity of the statements appearing in the internal proof, this process can substantially lengthen the proof.

In the case of the new proof of Grothendieck's generic freeness lemma, we were lucky to obtain a one-page proof using this process. (That the resulting external proof was so short is because it was possible to eliminate the use of the Noetherian property from the internal proof; without that elimination, the unwound proof was much longer.)

# Understanding algebraic geometry

Understand **notions of algebraic geometry** over a scheme $X$ as **notions of algebra** internal to $\mathrm{Sh}(X)$.

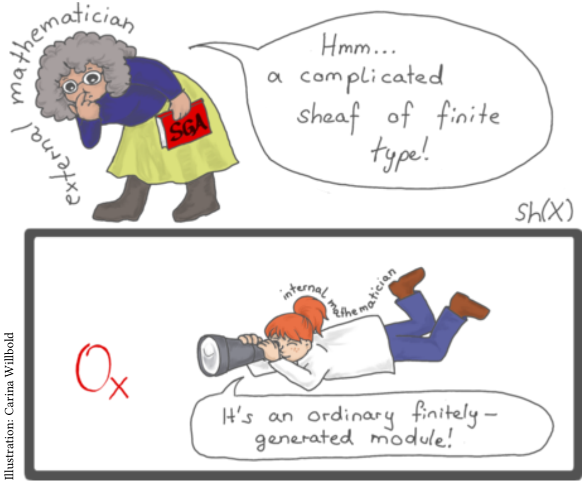


Illustration: Carina Willbold



One doesn't need to be an expert in topos theory in order to know that many notions in algebraic geometry are inspired by notions in algebra and that proofs in algebraic geometry often proceed by reducing to algebra. The internal language is a way of making this connection precise: In many cases, the former are simply interpretations of the latter internal to $\mathrm{Sh}(X)$. Because this connection is precise instead of informal, additional value is gained: We can skip many basic proofs in algebraic geometry because they're just externalizations of proofs in algebra carried out internally to $\mathrm{Sh}(X)$.

# Understanding algebraic geometry

Understand **notions of algebraic geometry** over a scheme $X$ as **notions of algebra** internal to $\mathrm{Sh}(X)$.

| externally | internally to $\mathrm{Sh}(X)$ |
|---|---|
| sheaf of sets | set |
| sheaf of modules | module |
| finite locally free sheaf | finite free module |
| tensor product of sheaves | tensor product of modules |
| sheaf of rational functions | total quotient ring of $\mathcal{O}_X$ |
| dimension of $X$ | Krull dimension of $\mathcal{O}_X$ |
| spectrum of a sheaf of $\mathcal{O}_X$-algebras | ordinary spectrum [with a twist] |
| higher direct image | sheaf cohomology |

Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of sheaves of $\mathcal{O}_X$-modules. If $\mathcal{F}'$ and $\mathcal{F}''$ are of finite type, so is $\mathcal{F}$.

$\Longleftarrow$

Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of modules. If $M'$ and $M''$ are finitely generated, so is $M$.



A basic example is as follows. A short exact sequence of sheaves of modules looks like a short exact sequence of plain modules from the internal point of view of $\mathrm{Sh}(X)$. If the two outer sheaves are of finite type, then from the internal point of view, the two outer modules will look like finitely generated modules. Because the standard proof of the proposition quoted on the lower right is constructively valid, it follows that, from the internal point of view, the middle module is too finitely generated. Consulting the dictionary a second time, this amounts to saying that the middle sheaf is of finite type.

A more advanced example is: The theorem “any finitely generated vector space does *not not* have a basis” of constructive linear algebra entails, by interpretation in $\mathrm{Sh}(X)$, that any sheaf of finite type over a reduced scheme is finite locally free on a *dense open subset.*

More details on this research program can be found in these notes, partly reported on at the 2015 IHÉS topos theory conference. Even though many important dictionary entries are still missing (for instance pertaining to derived categories and intersection theory), I believe that it is already in its current form useful to working algebraic geometers.

The next slide illustrates a further, different way of approaching algebraic geometry using topos theory.

# Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

- locally ringed spaces

  $$\text{set of prime ideals of } \mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n) + \\ \text{Zariski topology} + \text{structure sheaf}$$

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring $\to$ Set.

  $$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\}$$



At the Secret Blogging Seminar, there was an insightful long-running discussion on the merits of the two approaches. Two disadvantages of the approach using locally ringed spaces is that the underlying topological spaces don't actually parametrize "honest", "geometric" points, but the more complex notion of irreducible closed subsets; and that they don't work well in a constructive setting. (For this, they would have to be replaced by locally ringed locales.)

The functorial approach is more economical, philosophically rewarding, and works constructively. Given a functor $F$ : Ring $\to$ Set, we imagine $F(A)$ to be the set of "$A$-valued points" of the hypothetical scheme described by $F$, the set of "points with coordinates in $A$". These sets have direct geometric meaning. However, typically only field-valued points are easy to describe. For instance, the functor representing projective $n$-space is given on fields by

$$\begin{aligned} K \longmapsto &\text{ the set of lines through the origin in } K^{n+1} \\ &\cong \{[x_0 : \cdots : x_n] \mid x_i \neq 0 \text{ for some } i\}, \end{aligned}$$

whereas on general rings it is given by

$$\begin{aligned} A \longmapsto &\text{ the set of quotients } A^{n+1} \twoheadrightarrow P, \text{ where } P \text{ is projective}, \\ &\text{modulo isomorphism.} \end{aligned}$$

It is these more general kinds of points which impart a sense of cohesion on the field-valued points, so they can't simply be dropped from consideration.

# Synthetic algebraic geometry

Usual approach to algebraic geometry: **layer schemes above ordinary set theory** using either

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  set of prime ideals of $\mathbb{Z}[X, Y, Z]/(X^n + Y^n - Z^n)$ + Zariski topology + structure sheaf

- or Grothendieck's functor-of-points account, where a scheme is a functor Ring → Set.

$$A \longmapsto \{(x, y, z) \in A^3 \mid x^n + y^n - z^n = 0\}$$

**Synthetic approach:** model schemes **directly as sets** in the internal universe of the **big Zariski topos** of a base scheme.

$$\{(x, y, z) : (\underline{\mathbb{A}}^1)^3 \mid x^n + y^n - z^n = 0\}$$



This tension is resolved by observing that the category of functors Ring → Set is a topos (the *big Zariski topos* of Spec($\mathbb{Z}$)) and that we can therefore employ its internal language. This language takes care of juggling stages behind the scenes. For instance, projective $n$-space can be described by the naive expression

$$\{(x_0, \ldots, x_n) : (\underline{\mathbb{A}}^1)^{n+1} \mid x_0 \neq 0 \vee \cdots \vee x_n \neq 0\}/(\underline{\mathbb{A}}^1)^{\times}.$$

This example illustrates the goal: to develop a synthetic account of algebraic geometry, in which schemes are plain sets and morphisms between schemes are maps between those sets. It turns out that there are many similarities with the well-developed synthetic account of differential geometry, but also important differences, and it also turns out that synthetic algebraic geometry has close connections to a certain age-old unsolved problem in topos theory, the *mystery of nongeometric sequents*.

Details are in this set of slides.