Harmonic Analysis

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Contents

1 Lecturer Information

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2 Required Reading

1. Folland, G.B.: Fourier Analysis and its applications, Wadsworth & Brooks/Cole mathematics series, 1992

3 Additional Reading

1. Katznelson, Yitzhak. An introduction to Harmonic analysis. Cambridge University Press, 2004.

Part I

Basic Definitions and Theorems

1 Sequences and Series

Definition 1 (Convergent series)**.** The series $\sum\limits_{n=0}^{\infty} a_n$ is said to converge if the sequence of partial sums $S_N = \sum\limits_{n=0}^{N} a_n$ converges to a finite limit.

Definition 2 (Pointwise convergence of sequence of functions)**.** Let $D \subseteq \mathbb{R}$, and $\{f_n(x) : D \to \mathbb{R}\}$ be a sequence of functions. $f_n(x)$ is said to converge pointwise, to a limit function $f(x)$ on D, if $\forall \varepsilon > 0$, $\forall x \in D$, $\exists N \in \mathbb{N}$, such that $\forall n > N$, $|f_n(x) - f(x)| < \varepsilon$.

Definition 3 (Uniform convergence of sequence of functions)**.** Let $D \subseteq \mathbb{R}$, and $\{f_n(x) : D \to \mathbb{R}\}$ be a sequence of functions. $f_n(x)$ is said to converge uniformly to $f(x)$ on D if $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, such that, $\forall n > N$, $\forall x \in D$, $|f_n(x) - f(x)| < \varepsilon$.

Theorem 1. *If $\{f_n(x)\}_{n=1}^{\infty}$ are continuous functions, and $f_n(x) \xrightarrow{U} f(x)$, then $f(x)$ is also continuous.*

Theorem 2. *If a sequence of functions converges pointwise as well as uniformly, then the limit function must be the same.*

Theorem 3 (Weierstrass M-test)**.** *If $|u_k(x)| \leq c_k$ on D for $k \in \{1, 2, 3, \dots\}$ and the numerical series $\sum\limits_{k=1}^{\infty} c_k$ converges, then the series of functions $\sum\limits_{k=1}^{\infty} u_k(x)$ converges uniformly on D.*

2 Periodic Functions

Definition 4 (Periodic functions)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic if $\exists 0 < L \in \mathbb{R}$, such that $\forall x \in \mathbb{R}$,

$$f(x) = f(x + L)$$

If there exists a minimum L, it is called L^*, the fundamental period.

For a function $f(x) = k$, as any positive number is a period, there is no minimum L. Hence, $\nexists L^*$.

3 Odd and Even Functions

Definition 5 (Odd functions)**.** A function is said to be odd if $f(-x) = -f(x)$.

Odd functions are symmeteric about the origin.

Definition 6 (Even functions)**.** A function is said to be even if $f(-x) = f(x)$.

Odd functions are symmeteric about the y-axis.

Theorem 4. *If $h(x)$ is odd,*

$$\int\limits_{-L}^{L} h(x)\,\mathrm{d}x = 0$$

Part II
Introduction to Fourier Series

1 Real Fourier Series

Definition 7. Let $f : [-L, L] \to \mathbb{R}$, where $L > 0$. If $\forall x \in [-L, L]$, then

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right)$$

Theorem 5. *Let $L > 0$, $m \in \mathbb{W}$, $n \in \mathbb{W}$.*
Then

$$\int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \neq 0 \\ 2L & ; \quad m = n = 0 \end{cases}$$

Proof.

$$E = \int_{-L}^{L} \cos\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi}{L}x\right) \mathrm{d}x$$

$\because \cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)}{2} + \frac{\cos(\alpha-\beta)}{2}$

$$= \int_{-L}^{L} \frac{1}{2} \left(\cos\left((m+n)\frac{\pi}{L}x\right) + \cos\left((m-n)\frac{\pi}{L}x\right) \right) \mathrm{d}x$$

If $m \neq n$,

$$E = \frac{1}{2} \left(\frac{\sin\left((m+n)\frac{\pi}{L}x\right)}{(m+n)\frac{\pi}{L}} + \frac{\sin\left((m-n)\frac{\pi}{L}x\right)}{(m-n)\frac{\pi}{L}} \right) \Bigg|_{-L}^{L}$$

$$= 0$$

If $m = n \neq 0$,

$$E = \int_{-L}^{L} \frac{1}{2} \left(\cos\left(2m\frac{\pi}{L}x\right) + 1 \right) \mathrm{d}x$$

$$= \frac{1}{2} \int_{-L}^{L} \cos\left(2m\frac{\pi}{L}x\right) \mathrm{d}x + \frac{1}{2}x \Big|_{-L}^{L}$$

$$= L$$

If $m = n = 0$,

$$
\begin{aligned}
E &= \int_{-L}^{L} \cos(0)\cos(0)\,\mathrm{d}x \\
&= x|_{-L}^{L} \\
&= 2L
\end{aligned}
$$

□

Theorem 6. *Let $L > 0$, $m \in \mathbb{N}$, $n \in \mathbb{N}$.*
Then

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x = \begin{cases} 0 & ; \quad m \neq n \\ L & ; \quad m = n \end{cases}$$

Theorem 7. *Let $L > 0$, $m \in \mathbb{W}$, $n \in \mathbb{W}$.*
Then

$$\int_{-L}^{L} \sin\left(\frac{m\pi}{L}x\right)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x = 0$$

Assuming $f(x)$ is known, and assuming that it can be integrated term by term,

$$
\begin{aligned}
\int_{-L}^{L} f(x)\,\mathrm{d}x &= \int_{-L}^{L} \frac{1}{2}a_0\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-L}^{L} \cos\left(n\frac{\pi}{L}x\right)\mathrm{d}x + b_n \int_{-L}^{L} \sin\left(n\frac{\pi}{L}x\right)\mathrm{d}x \\
\therefore \int_{-L}^{L} f(x)\,\mathrm{d}x &= \frac{1}{2}\int_{-L}^{L} a_0\,\mathrm{d}x \\
&= \frac{1}{2}a_0 \cdot 2L \\
\therefore a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,\mathrm{d}x
\end{aligned}
$$

Similarly, multiplying the series with $\cos\left(m\frac{\pi}{L}x\right)$ for $m \neq 0$ and integrating,

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(m\frac{\pi}{L}x\right)\mathrm{d}x$$

for $m \in \mathbb{N}$.
Similarly, for $m \in \mathbb{N} \setminus \{0\}$,

$$b_m = \frac{1}{L} \int\limits_{-L}^{L} f(x) \sin\left(m \frac{\pi}{L} x\right) \mathrm{d}x$$

Definition 8. The expansion

$$f(x) \approx \frac{1}{2} a_0 + \sum_{i=1}^{\infty} \left(a_n \cos\left(n \frac{\pi}{L} x\right) + b_n \sin\left(n \frac{\pi}{L} x\right)\right)$$

where, for $m \in \mathbb{N}$,

$$a_m = \frac{1}{L} \int\limits_{-L}^{L} f(x) \cos\left(m \frac{\pi}{L} x\right) \mathrm{d}x$$

and, for $m \in \mathbb{N} \setminus \{0\}$,

$$b_m = \frac{1}{L} \int\limits_{-L}^{L} f(x) \sin\left(m \frac{\pi}{L} x\right) \mathrm{d}x$$

is called the Fourier Series of $f(x)$.

2 Complex Fourier Series

By Euler's formula,

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$
$$\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$$

Therefore,

$$\frac{1}{2i} = -\frac{i}{2}$$

Therefore, substituting in the Fourier series,

$$
\begin{aligned}
f(x) &\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\frac{1}{2}\left(e^{\frac{in\pi}{L}x}+e^{-\frac{in\pi}{L}x}\right)-b_n\frac{i}{2}\left(e^{\frac{in\pi}{L}x}-e^{-\frac{in\pi}{L}x}\right)\right)\\
&\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(e^{\frac{in\pi}{L}x}\left(\frac{1}{2}a_n-\frac{i}{2}b_n\right)+e^{-\frac{in\pi}{L}x}\left(\frac{1}{2}a_n+\frac{i}{2}b_n\right)\right)\\
&\approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(e^{\frac{in\pi}{L}x}\left(\frac{1}{2}a_n-\frac{i}{2}b_n\right)\right)+\sum_{n=-\infty}^{1}\left(e^{\frac{in\pi}{L}x}\left(\frac{1}{2}a_n+\frac{i}{2}b_n\right)\right)\\
&\approx \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi}{L}x}
\end{aligned}
$$

3 Bessel's Inequality

Definition 9 (Piecewise continuous functions)**.** $f:\mathbb{R}\to\mathbb{R}$ is said to be piecewise continuous if, for every finite interval $[a,b]$ there is a finite number of discontinuity points, and the one-sided limits at each of these points are also finite.

Definition 10 (Piecewise continuously differentiable functions)**.** $f:\mathbb{R}\to\mathbb{R}$ is said to be piecewise continuously differentiable if it is piecewise continuous, and

$$\lim_{h\to 0^+}\frac{f(x+h)-f(x^+)}{h}<\infty$$

and

$$\lim_{h\to 0^-}\frac{f(x+h)-f(x^-)}{h}<\infty$$

Theorem 8 (Bessel's Inequality)**.** *Let $f(x)$ be a piecewise continuous function defined on $[-L,L]$. Then*

$$\frac{1}{2}{a_0}^2+\sum_{n=1}^{\infty}{a_n}^2+{b_n}^2\le\frac{1}{L}\int_{-L}^{L}f(x)^2\,\mathrm{d}x$$

4 Riemann-Lebesgue's Lemma

Theorem 9 (Riemann-Lebesgue's Lemma)**.** *If $f(x)$ is piecewise continuous on $[-L,L]$, then*

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=0$$

Proof. By Bessel's Inequality,

$$\frac{1}{2}{a_0}^2 + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2 \leq \int\limits_{-L}^{L} f(x)^2 \, \mathrm{d}x$$

$$\therefore \frac{1}{2}{a_0}^2 + \sum_{n=1}^{\infty} {a_n}^2 + {b_n}^2 < \infty$$

As the function is piecewise continuous in $[-L, L]$, its integral from $-L$ to L is finite.

Therefore,

$$\lim_{n\to\infty} {a_n}^2 \leq \lim_{n\to\infty} {a_n}^2 + {b_n}^2$$

$$\therefore \lim_{n\to\infty} \leq 0$$

Therefore,

$$\lim_{n\to\infty} a_n = 0$$

Similarly,

$$\lim_{n\to\infty} b_n = 0$$

□

Exercise 1.
If $f(x)$ is piecewise continuous on $[-\pi, \pi]$, show that

$$\lim_{n\to\infty} \int\limits_{-\pi}^{\pi} f(x) \sin\left(\left(n + \frac{1}{2}\right) x\right) \mathrm{d}x = 0$$

Solution 1.

$$\sin\left(\left(n + \frac{1}{2}\right) x\right) = \sin(nx)\cos\left(\frac{1}{2}x\right) + \cos(nx)\sin\left(\frac{1}{2}x\right)$$

Therefore, the limit is

$$\begin{aligned} 0 = &\lim_{n\to\infty} \int\limits_{-\pi}^{\pi} f(x) \cos\left(\frac{1}{2}x\right) \sin(nx) \, \mathrm{d}x \\ &+ \lim_{n\to\infty} \int\limits_{-\pi}^{\pi} f(x) \sin\left(\frac{1}{2}x\right) \cos(nx) \, \mathrm{d}x \end{aligned}$$

Let

$$g_1 = f(x)\cos\left(\frac{1}{2}x\right)$$
$$g_2 = f(x)\sin\left(\frac{1}{2}x\right)$$

Therefore,

$$\lim_{n\to\infty}\left(\pi b_n(g_1) + \pi a_n(g_2)\right) = 0$$

Therefore,

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin\left(\left(n+\frac{1}{2}\right)x\right)\mathrm{d}x = 0$$

5 Dirichlet's Kernel

Definition 11 (Dirichlet kernel)**.**

$$D_m(t) = \frac{1}{2} + \sum_{n=1}^{m}\cos(nt)$$

is called the Dirichlet kernel of order m.

Theorem 10 (Second representation of Dirichlet's kernel)**.** *Let $m \in \mathbb{N}$. Then, for $t \neq 2\pi k$, where $k \in \mathbb{Z}$,*

$$\begin{aligned} D_m(t) &= \frac{1}{2} + \cos(t) + \cos(2t) + \cdots + \cos(mt) \\ &= \frac{\sin\left(\left(m+\frac{1}{2}\right)t\right)}{2\sin\left(\frac{1}{2}t\right)} \end{aligned}$$

Theorem 11. *Let*

$$S_m(f,x) = \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n\cos(nx) + b_n\sin(nx)$$

Then,

$$S_m(f,x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\left(\frac{1}{2}\sum_{n=1}^{m}\cos(nt)\right)\mathrm{d}t$$

Proof.

$$
\begin{aligned}
S_m(f,x) &= \frac{1}{2}a_0 + \sum_{n=1}^{m} a_n \cos(nx) + b_n \sin(nx) \\
&= \quad \frac{1}{2}\underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\,\mathrm{d}s\right)}_{a_0} \\
&\quad + \sum_{n=1}^{m} \underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\cos(ns)\,\mathrm{d}s\right)}_{a_n} \cos(nx) \\
&\quad + \sum_{n=1}^{m} \underbrace{\left(\frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\sin(ns)\,\mathrm{d}s\right)}_{b_n} \sin(nx) \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left(\frac{1}{2} + \sum_{n=1}^{m} \cos(ns)\cos(nx) + \sin(ns)\sin(nx)\right)\mathrm{d}s \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(s)\left(\frac{1}{2} + \sum_{n=1}^{m} \cos\left(n(s-x)\right)\right)\mathrm{d}s
\end{aligned}
$$

Let

$$
\begin{aligned}
t &= s - x \\
\therefore \mathrm{d}t &= \mathrm{d}s
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_m(f,x) &= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} f(t+x)\left(\frac{1}{2} + \sum_{n=1}^{m} \cos(nt)\right)\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)D_m(t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t+x)D_m(-t)\,\mathrm{d}t \\
&= \frac{1}{\pi}\left(f(t) * D_m(t)\right)
\end{aligned}
$$

As the function is 2π-periodic, the limits can be changed from $-\pi - x$ and $\pi - x$ to $-\pi$ and π.

$\square$

Theorem 12 (Dirichlet Theorem). *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a piecewise continuously differentiable function.*
Then, $\forall x \in (-\pi, \pi)$,

This theorem is also valid for $[-L, L]$.

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(x^-) + f(x^+)}{2}$$

and for $x = \pi$ or $x = -\pi$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) = \frac{f(\pi^-) + f(-\pi^+)}{2}$$

Exercise 2.
Prove that $\forall x \in [0, 1]$,

$$x(\pi - x) = \frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{1}{n^2} \cos(2nx)$$

Solution 2.
Let the function be extended naturally to $[0, \pi]$. Hence, let the function be extended evenly to $[-\pi, \pi]$.
Therefore as the function is even, the Fourier series of the function is

$$x(\pi - x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

Therefore,

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x \\ &= \frac{1}{\pi} \int_{0}^{\pi} x(\pi - x)\,\mathrm{d}x \\ &= \frac{\pi^2}{3} \end{aligned}$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} x(\pi - x)\cos(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\int_{0}^{\pi} \left(x\pi - x^2\right)\cos(nx)\,\mathrm{d}x \\
&= \frac{2}{\pi}\left(\left(x\pi - x^2\right)\int \cos(nx)\,\mathrm{d}x - \int (\pi - 2x)\int \cos(nx)\,\mathrm{d}x\,\mathrm{d}x\right)\Bigg|_0^{\pi}
\end{aligned}
$$

The integral of $\cos x$ from 0 to π is zero, i.e. if the limits are π and 0, the function $\sin x$ is zero.

$$
\begin{aligned}
&= \frac{2}{\pi}\left(\left(x\pi - x^2\right)\frac{\cancelto{0}{\sin(nx)}}{n} - \int (\pi - 2x)\frac{\sin(nx)}{n}\,\mathrm{d}x\right)\Bigg|_0^{\pi} \\
&= \frac{2}{\pi}\left(-\int (\pi - 2x)\frac{\sin(nx)}{n}\,\mathrm{d}x\right)\Bigg|_0^{\pi}
\end{aligned}
$$

The integral of $\cos x$ from 0 to π is zero.

$$
\begin{aligned}
&= \frac{2}{\pi}\left((\pi - 2x)\frac{\cos(nx)}{n^2} + \int \frac{2\cos(nx)}{n^2}\,\mathrm{d}x\right)\Bigg|_0^{\pi} \\
&= \frac{2}{\pi}\,(\pi - 2x)\frac{\cos(nx)}{n^2}\Bigg|_0^{\pi} \\
&= \frac{2}{n^2}\left((-1)^{n+1} - 1\right)
\end{aligned}
$$

Therefore,

$$
a_n = \begin{cases} -\frac{4}{n^2} & ; \quad n = 2k \\ 0 & ; \quad n = 2k+1 \end{cases}
$$

Therefore,

$$
x(\pi - x) = \frac{\pi^2}{6} - \sum_{k=1}^{\infty} \frac{1}{n^2}\cos(2\pi k)
$$

Theorem 13. *Let $f[-\pi, \pi] \to \mathbb{R}$ be continuous and $f(-\pi) = f(\pi)$. Let $f'(x)$ be piecewise continuous. Then the Fourier series converges absolutely to some limit and uniformly to $f(x)$.*

6 Relation between Fourier Coefficients of $f(x)$ and $f'(x)$

Theorem 14. *Let the Fourier coefficients of $f(x)$ be a_0, a_n, and b_n. Then, the Fourier coefficients of $f'(x)$ are*

$$\alpha_0 = 0$$
$$\alpha_n = nb_n$$
$$\beta_n = -na_n$$

Proof. Assuming $f'(x)$ is integrable,

$$f'(x) \approx \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos(nx) + \beta_n \sin(nx)$$

Therefore,

$$\begin{aligned} \alpha_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\,\mathrm{d}x \\ &= \frac{f(\pi) - f(-\pi)}{\pi} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \alpha_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f'(x)\cos(nx)\,\mathrm{d}x \\ &= \frac{1}{\pi}\, f(x)\cos(nx)|_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \end{aligned}$$

Therefore,

$$\alpha_n = nb_n$$
$$\beta_n = -na_n$$

□

Theorem 15. *Let the complex Fourier coefficient of $f(x)$ be c_n. Then, the complex Fourier coefficient of $f'(x)$ is*

$$\gamma_n = inc_n$$

On a general interval, this theorem translates to term-by-term differentiation, i.e., the order of summation and differentiation can be changed.

Theorem 16. *Let $f(x) : [-\pi, \pi] \to \mathbb{R}$ be a continuous function such that $f(-\pi) = f(\pi)$, and let $f'(x)$ be piecewise continuous.*
If

$$
\begin{aligned}
f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx) \\
&= \int_{n=\infty}^{\infty} c_n e^{inx}
\end{aligned}
$$

then,

$$
\begin{aligned}
f'(x) &\approx \sum_{n=1}^{\infty} nb_n \cos(nx) - na_n \sin(nx) \\
&= \sum_{n=-\infty}^{\infty} inc_n e^{inx}
\end{aligned}
$$

This theorem also holds for a general interval $[-L, L]$.

Theorem 17. *Let $f : [-\pi, \pi] \to \mathbb{R}$ be a continuous function that maintains $f(-\pi) = f(\pi)$ and let $f'(x)$ be piecewise continuous. Then, $S_m(f, x)$ converges uniformly to $f(x)$.*

Definition 12 (Inner product)**.** Let $\overline{x}$ and $\overline{y}$ be vectors. Their inner product is defined to be

$$\langle \overline{x}, \overline{y} \rangle = \sum_{i=1}^{\infty} x_i y_i$$

Theorem 18.

$$|\langle \overline{x}, \overline{y} \rangle| \leq \sqrt{\langle \overline{x}, \overline{x} \rangle} \sqrt{\langle \overline{y}, \overline{y} \rangle}$$

Theorem 19. *Let $f(x)$ be continuous on $[-\pi, \pi]$ with piecewise continuous $f'(x)$. Let $S_m(f, x)$ converge uniformly to $f(x)$. Then, the Fourier series is term-by-term differentiable.*