Reparametrization of COM-Poisson Regression Models with Applications in the Analysis of Experimental Count Data

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Outline

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1

Background

Count data

Number of times an event occurs in the observation unit.

Random variables that assume non-negative integer values.

Let Y be a counting random variable, so that $y = 0, 1, 2, \dots$

Examples in experimental researches:

- number of grains produced by a plant;
- number of fruits produced by a tree;
- number of insects on a particular cell;
- others.

Poisson model and limitations

GLM framework (NELDER; WEDDERBURN, 1972)

- Provide suitable distribution for a counting random variables;
- Efficient algorithm for estimation and inference;
- Implemented in many software.

Poisson model

- Relationship between mean and variance, $E(Y) = \text{var}(Y)$;

Main limitations

- Overdispersion (more common), $E(Y) < \text{var}(Y)$
- Underdispersion (less common), $E(Y) > \text{var}(Y)$

COM-Poisson distribution

- Probability mass function (SHMUELI et al., 2005) takes the form

$$\Pr(Y = y \mid \lambda, \nu) = \frac{\lambda^y}{(y!)^\nu Z(\lambda, \nu)}\,, \qquad Z(\lambda, \nu) = \sum_{j=0}^{\infty} \frac{\lambda^j}{(j!)^\nu}\,, \tag{1}$$

where $\lambda > 0$ and $\nu \geq 0$.

- Moments are not available in closed form;
- Expectation and variance can be closely approximated by

$$E(Y) \approx \lambda^{1/\nu} - \frac{\nu - 1}{2\nu} \qquad \text{and} \qquad \text{var}(Y) \approx \frac{\lambda^{1/\nu}}{\nu}$$

with accurate approximations for $\nu \leq 1$ or $\lambda > 10^\nu$ (SHMUELI et al., 2005; SELLERS; BORLE; SHMUELI, 2012).

COM-Poisson regression models

Model definition

- Modelling the relationship between $E(Y_i)$ and $\boldsymbol{x}_i$ indirectly (SELLERS; SHMUELI, 2010);

$$
\begin{aligned}
Y_i \mid \boldsymbol{x}_i &\sim \text{COM-Poisson}(\lambda_i, \nu) \\
\eta(E(Y_i \mid \boldsymbol{x}_i)) &= \log(\lambda_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta}
\end{aligned}
$$

Main goal

- Propose a reparametrization in order to model the expectation of the response variable as a function of the covariate values directly.

2

Reparametrization

Reparametrized COM-Poisson

Reparametrization

- Introduced new parameter μ, using the mean approximation

$$\mu = \lambda^{1/\nu} - \frac{\nu - 1}{2\nu} \quad \Rightarrow \quad \lambda = \left(\mu + \frac{(\nu - 1)}{2\nu}\right)^{\nu};$$

- Precision parameter is taken on the log scale to avoid restrictions on the parameter space

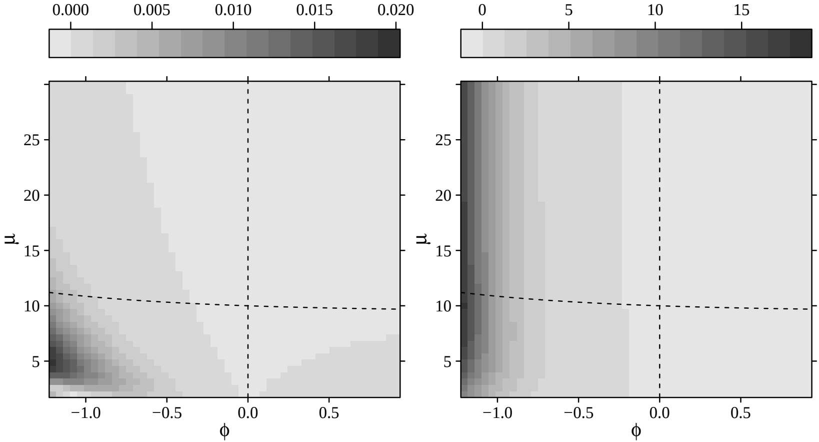
$$\phi = \log(\nu) \Rightarrow \phi \in \mathbb{R}$$

Probability mass function

- Replacing λ and ν as function of μ and ϕ in Equation 1

$$\Pr(Y = y \mid \mu, \phi) = \left(\mu + \frac{e^{\phi} - 1}{2e^{\phi}}\right)^{ye^{\phi}} \frac{(y!)^{-e^{\phi}}}{Z(\mu, \phi)}.$$

Study of the moments approximations



(a) Errors for mean approximation

(b) Errors for variance approximation

Figure: Quadratic errors for moments approximation. Dotted lines representing the restriction for good approximations by (SHMUELI et al., 2005).

COM-Poisson$_\mu$ distribution

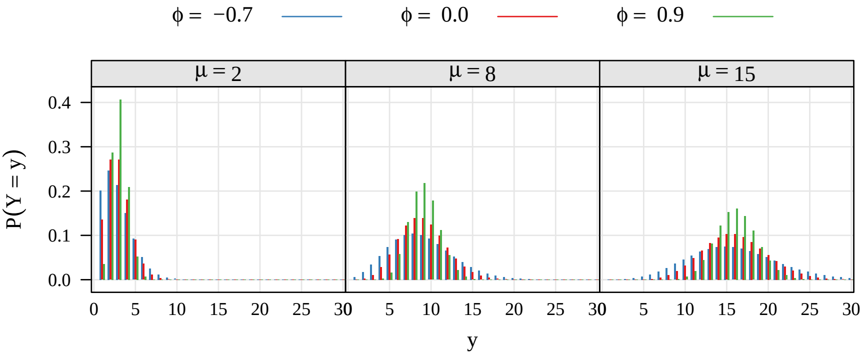


Figure: Shapes of the COM-Poisson distribution for different parameter values.

COM-Poisson$_\mu$ regression models

Model definition

- Modelling relationship between $E(Y_i)$ and $\boldsymbol{x}_i$ directly

$$Y_i \mid \boldsymbol{x}_i \sim \text{COM-Poisson}_\mu(\mu_i, \phi)$$
$$\log(E(Y_i \mid \boldsymbol{x}_i)) = \log(\mu_i) = \boldsymbol{x}_i^\top \boldsymbol{\beta}$$

Estimation and Inference

- Parameter estimates are obtained by numerical maximization of the log-likelihood function (by BFGS algorithm);
- Standard errors for regression coefficients are obtained based on the observed information matrix;
- Confidence intervals for $\hat{\mu}_i$ are obtained by delta method.

3

Case studies

Artificial defoliation in cotton phenology

Aim: to assess the effects of five defoliation levels on the bolls produced at five growth stages;

Design: factorial 5×5, with 5 replicates;

Experimental unit: a plot with 2 plants;

Factors:

- Artificial defoliation (`des`): 0, 0.25, 0.5, 0.75, 1
- Growth stage (`est`): vegetative, flower bud, blossom, fig, cotton boll

Response variable: Total number of cotton bolls;

Model specification

Linear predictor: following Zeviani et al. (2014)

- $\log(\mu_{ij}) = \beta_0 + \beta_{1j}\text{def}_i + \beta_{2j}\text{def}_i^2$
  i varies in the levels of artificial defoliation;
  j varies in the levels of growth stages.

Alternative models:

- Poisson (μ_{ij});
- COM-Poisson ($\lambda_{ij} = \eta(\mu_{ij})$, ϕ)
- COM-Poisson$_\mu$ (μ_{ij} , ϕ)
- Quasi-Poisson ($\text{var}(Y_{ij}) = \sigma\mu_{ij}$)

Parameter estimates

Table: Parameter estimates (Est) and ratio between estimate and standard error (SE)

	Poisson		COM-Poisson		COM-Poisson$_\mu$		Quasi-Poisson	
	Est	Est/SE	Est	Est/SE	Est	Est/SE	Est	Est/SE
ϕ, σ			1.585	12.417	1.582	12.392	0.241	
β_0	2.190	34.572	10.897	7.759	2.190	74.640	2.190	70.420
β_{11}	0.437	0.847	2.019	1.770	0.435	1.819	0.437	1.726
β_{12}	0.290	0.571	1.343	1.211	0.288	1.223	0.290	1.162
β_{13}	-1.242	-2.058	-5.750	-3.886	-1.247	-4.420	-1.242	-4.192
β_{14}	0.365	0.645	1.595	1.298	0.350	1.328	0.365	1.314
β_{15}	0.009	0.018	0.038	0.035	0.008	0.032	0.009	0.036
β_{21}	-0.805	-1.379	-3.725	-2.775	-0.803	-2.961	-0.805	-2.809
β_{22}	-0.488	-0.861	-2.265	-1.805	-0.486	-1.850	-0.488	-1.754
β_{23}	0.673	0.989	3.135	2.084	0.679	2.135	0.673	2.015
β_{24}	-1.310	-1.948	-5.894	-3.657	-1.288	-4.095	-1.310	-3.967
β_{25}	-0.020	-0.036	-0.090	-0.076	-0.019	-0.074	-0.020	-0.074
LogLik	-255.803		-208.250		-208.398		—	
AIC	533.606		440.500		440.795		—	
BIC	564.718		474.440		474.735		—	

Fitted curves

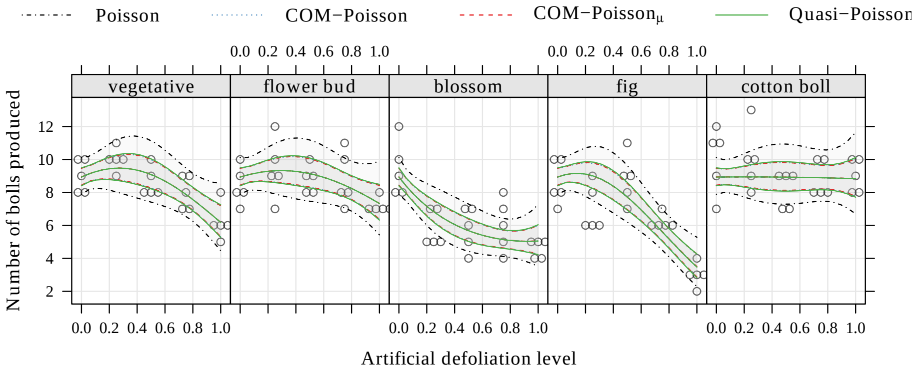


Figure: Curves of fitted values with 95% confidence intervals.

Additional results

- Empirical correlations between $\hat{\phi}$ and $\hat{\beta}$ estimators is approximately 0 for reparametrized model.

Table: Empirical correlations between dispersion and location parameters estimators.

	$\hat{\beta}_0$	$\hat{\beta}_{11}$	$\hat{\beta}_{12}$	$\hat{\beta}_{13}$	$\hat{\beta}_{14}$	$\hat{\beta}_{15}$	$\hat{\beta}_{21}$	$\hat{\beta}_{22}$	$\hat{\beta}_{23}$	$\hat{\beta}_{24}$	$\hat{\beta}_{25}$
COM-Poisson	0.995	0.223	0.153	-0.490	0.161	0.004	-0.350	-0.228	0.263	-0.458	-0.009
COM-Poisson$_\mu$	0.001	-0.000	-0.000	-0.001	-0.001	-0.000	0.000	0.000	0.001	0.002	0.000

- COM-Poisson fit was 34.347% slower than COM-Poisson$_\mu$;

4

Final remarks

Concluding remarks

Summary

- Over/under-dispersion needs caution;
- COM-Poisson is a suitable choice for these situations;
- The proposed reparametrization, COM-Poisson$_\mu$ has some advantages:
  - Simple transformation of the parameter space;
  - Full parametric approach;
  - Correlation between the estimators was practically null;
  - Faster for fitting;
  - Allows interpretation of the coefficients directly (like GLM-Poisson model).

Future work

- Simulation study to assess model robustness against distribution miss specification;
- Assess theoretical approximations for $Z(\lambda, \nu)$ (or $Z(\mu, \phi)$), in order to avoid the selection of sum's upper bound;
- Propose a double GLM based on the COM-Poisson$_\mu$ model.

- Full-text article is available on ResearchGate (in portuguese) <https://www.researchgate.net/publication/316880329>
- All codes (in R) and source files are available on GitHub <https://github.com/jreduardo/rbras2017>

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